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Coherent interfaces with junctions in continua with microstructure

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Dedicated to my father, Luigi Mariano, painter, artist

Abstract

In the present paper, the influence of material substructures and of the inertia on the evolution of junctions among coherent discontinuity surfaces is discussed within the setting of multifield theories. An evolution equation for the junction is deduced in addition to the ones for the interfaces. The presence of substructures (or microstructures) within the body is evaluated through order parameters. Substructural interactions are considered and need to be balanced. The case in which such interactions can be decomposed into self-forces and microstresses is dealt with. Finally, the interfaces are endowed by peculiar structure: line stresses and microstresses are considered, as well as line substructural self-forces. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Preliminary remarks

Junctions among coherent interfaces are recurring phenomena in solids. Grains meet each other; mixtures of immiscible fluids (e.g., the suspension of oil in water) or of different solid phases exhibit junctions. Typical examples in solids are X-interfaces between austenite and martensite, [1] or junctions arising in long-period superstructures in Al_5Ti_3, r-Al_2Ti and Al_3Ti ($L1_2$-type) alloys (e.g., Nakano et al., 1999). In metals, at junctions, atoms are crowded in a geometry in which the minimum of the energy is greater than the one on the geometry of the bulk. So, junctions are endowed by an excess of energy (a line energy, in three-dimensional bodies). Such an energy, free energy, may be evaluated experimentally, by using calorimetry devices, or numerically. When phase-transitions occur within the body, the junctions move according to the evolution of the interfaces determining them. The material surrounding the interface exerts a force on it, which can drive or obstruct the interface itself. Such an interaction, called also configurational force, is

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[1] See Ruddock (1994) for the theoretical questions arising in the case of presence of X-interfaces.

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expressed through Eshelby's tensor (Eshelby, 1975) whose validity in non-conservative setting has been proven by Gurtin (1995) [2] in the case of simple material. Gurtin's argument is substantially based on the requirement of the invariance of the power under re-parameterization of the interface and on an appropriate version of the second law.

In 1999, I indicated some elementary corollaries to Gurtin's theory within the setting of multifield theories describing the influence of the microstructure (or substructure) of the material on the gross mechanical behavior of the body (Mariano, 2000). The presence of such substructures is evaluated by assigning order parameters to each material patch. They are elements of some manifold $\mathscr{M}$ and represent the geometrical model of the substructure itself. Basically, they are considered as observable quantities. So, interactions must be associated to them. The representation of such interactions depends on the geometrical properties of $\mathscr{M}$, in particular on the possibility of defining on the manifold a physically significant connection (Capriz, 1985, 1989, 2000), i.e. a manner of constructing covariant gradients.

When such a connection exists, the substructural interactions may be decomposed into microstresses and self-forces. In this context, the explicit expression of Eshelby tensor must be augmented by an appropriate product of the covariant gradient of the order parameter with the microstress, and the result of the product is a second-order tensor, whatever the tensor rank of the order parameter is.

This modified Eshelby tensor may be a useful tool even for evaluating the behavior of interfaces or cracks within the setting of direct models of plates and shells.

An analogous result may be considered for the surface Eshelby stress. It can be defined when surface stress measures are defined on the discontinuity surfaces, considered like thin domains capable to suffer shear.

Simha and Bhattacharya (1998) have studied the expression of configurational forces acting on junctions in two-dimensional simple bodies. Their result includes terms representing limiting values at the junction of the interactions in bulk and interfaces surrounding the junction. Their work suggests to develop an analogous analysis within the context of multifield theories on the basis of the quoted modified expressions of bulk and interfacial Eshelby tensors.

This is the simple goal of the present paper. The influence of inertial effects is also studied.

As in Simha and Bhattacharya (1998), the two-dimensional case is considered. It is the simplest one and allows one to underline clearly the contribution of microstresses and self-forces (relevant to the material substructures) to the expression of configurational force driving the junction during phase transitions.

The result within the setting of multifield theories presented here allows to analyze, among other things, the behavior of plane junctions among interfaces in directed models of plates (see, e.g., Antman, 1995) made of materials undergoing phase transitions, as for example shape memory alloys.

The three-dimensional case will be presented later. Technically, it is more difficult than the two-dimensional one. The junction is line endowed not only by line tension and shear but also by an additional line microstress and line self-force. Such additional generalized line measures of interaction enter in the general expression of configurational forces acting at the junction.

Conceptually, the passage from the two-dimensional to the three-dimensional case implies another step in understanding the behavior of junctions. By inspection of the sole balance equations at the junction (involving both the line measures of interactions and the limiting contributions of bulk and surface ones), in fact, it is possible to prove the possibility for the line free energy to assume negative values. This is in accord to Gibbs' conjecture on the energetic behavior of junctions. This could also have consequences on the analysis of the stability of grain substructures at junctions.

[2] Detailed analyses of the applications of Eshelby tensor in various physical circumstances can be found in (Maugin and Trimarco, 1992, 1995).

Remarks on the notation: In the following, if A is an $(n+1)$-covariant and m-contravariant tensor and B is an n-covariant and $(m+1)$-contravariant tensor, the product $A * B$ furnishes a 1-covariant and 1-contravariant second-order tensor, i.e. $(A * B)\mathrm{v} = \mathrm{w}$, where v is a vector and w a covector. Moreover, in the case in which A is $(n+1)$-covariant and m-contravariant and B is n-covariant and m-contravariant tensor, the product $A \overset{\bullet}{*} B$ furnishes a covariant vector as result. Additionally, $\mathrm{w} * \mathrm{v} = \mathrm{w} \boxtimes \mathrm{v}$.

2. Configurations

As anticipated in Section 1, to each material patch $\mathbf{P}$, the pair $(\mathbf{x}, \varphi)$ is associated. $\mathbf{x}$ is the *placement* within the Euclidean space $\mathscr{E}$ (which is considered two-dimensional in the present context). φ is the *order parameter*, a n-covariant, m-contravariant tensor belonging to $\mathscr{M}$. It is the *geometrical* model of the material substructure. Typical examples are the following:

- $\varphi(\cdot)$ may be a scalar-valued field defined on the whole body. So, the order parameter may represent the void volume fraction in porous materials (Nunziato and Cowin, 1979), the volume fraction of a material specie in liquid or solid mixtures, and so on.
- φ may take values in the projective plane and be a representation of the orientation of straight molecules in nematic liquid crystals or in smectics (Ericksen, 1962; Capriz, 1995).
- φ may be a second-order symmetric tensor. So, it can represent the local deformation of big molecules (Mindlin, 1964) within an indistinct matrix or it can be the second-order approximation (the dipole one) of the microcrack density distribution in microcracked bodies (Mariano and Augusti, 1998).
- The whole range of direct models of structural elements such as plates, shells or rods (Antman, 1995; Villaggio, 1997; Ericksen and Truesdell, 1958) is a collection of special cases of multifield theories.

Throughout the paper, $\mathscr{M}$ will be considered finite dimensional and paracompact but, of course, models in which $\mathscr{M}$ is infinite dimensional are possible, as in the case in which, e.g., an entire distribution on some set of directions is assigned to each material patch.

In any case, with these premises, the physical configuration of the body B is given by mapping K defined by

$$K : \mathrm{B} \to \mathscr{E}^2 \times \mathscr{M}, \tag{2.1}$$

$$(\mathrm{B} \ni)\mathbf{P} \overset{K}{\mapsto} (\mathbf{x}, \varphi), \tag{2.2}$$

$K_{\mathscr{E}^2}(\mathrm{B}) \equiv \mathscr{B}$ is the *apparent configuration*. The mapping $K_{\mathscr{E}^2}$ is an homeomorphism.

Of course, for every $\mathbf{P}$ belonging to B, $\mathbf{x}(\mathbf{P}) \equiv K_{\mathscr{E}^2}(\mathbf{P}), \varphi(\mathbf{P}) \equiv K_{\mathscr{M}}(\mathbf{P})$. Given two arbitrary configurations, say K_1 and K_2, it is assumed that the mapping $K \equiv K_2 \circ K_1^{-1}$ is *continuous* and *piecewise continuously differentiable*.

Take, at this point, a reference apparent configuration $\mathscr{B}$ and consider it endowed by an atlas of coordinates $\{\mathbf{X}\}$. In particular, I assume that $\mathscr{B}$ is a finite union of fit regions in Noll's and Virga's sense; [3] namely $\mathscr{B}$ is the finite union of bounded sets of $\mathscr{E}^2$ which are regularly open [4] and possess finite perimeter of zero volume measure.

Given a point $\mathbf{X}$ on $\mathscr{B}$, its current placement is given by $\mathbf{x}(\mathbf{X}) \equiv K_{\mathscr{E}^2}(\mathbf{X})$, and it is assumed that $K_{\mathscr{E}^2}$ is *orientation preserving*, namely that $\det \mathbf{F} > 0$, where $\mathbf{F}$ is the gradient of deformation.

[3] See Noll and Virga (1988) for a detailed explanation of the principal properties and of the role of fit regions in mechanics.

[4] A set of the Euclidean space is called "regularly open" when it coincides with the interior of its closure.

A motion of the body will be described by fields

$$\mathbf{x}(\cdot,\cdot) : \mathscr{B} \times [0,d] \to \mathscr{E}^2, \tag{2.3}$$

$$\varphi(\cdot,\cdot) : \mathscr{B} \times [0,d] \to \mathscr{M} \tag{2.4}$$

in a way so that the current position of a particle $\mathbf{X} \in \mathscr{B}$ at time $t \in [0,d]$ is given by $\mathbf{x}(\mathbf{X},t)$, while the current value of the order parameter is $\varphi(\mathbf{X},t)$.

So, velocities $\dot{\mathbf{x}}$ and $\dot{\varphi}$ may be defined and Vel indicates the space of pairs $(\dot{\mathbf{x}},\dot{\varphi})$. Among all the possible velocity fields, the *rigid* ones play a special role in the following developments. In order to define them, consider a time-parameterized family of rotations represented through their characteristic vector, namely $\mathbf{q}(t)$, the rigid velocity $\dot{\mathbf{x}}_{\mathrm{rig}}$ of the placement $\mathbf{x}(\mathbf{X},t)$ is given by

$$\dot{\mathbf{x}}_{\mathrm{rig}} = \mathbf{c}(t) + \dot{\mathbf{q}} \times (\mathbf{x} - \mathbf{x}_{\mathbf{c}}), \tag{2.5}$$

where $\mathrm{c}(t)$ is the translation velocity and $\mathbf{x}_{\mathrm{c}}$ the position of the center of rotation.

After a rotation, the order parameter can be expressed by series expansion as

$$\varphi_{\mathbf{q}} = \varphi(\mathbf{X}) + \left.\frac{\mathrm{d}\varphi_{\mathbf{q}}}{\mathrm{d}\mathbf{q}}\right|_{\mathbf{q}=\mathbf{0}} \mathbf{q} + \mathrm{o}(|\mathbf{q}|^2). \tag{2.6}$$

Time derivative of Eq. (2.5) implies (neglecting higher order terms)

$$\dot{\varphi}_{\mathbf{q}} = \left.\frac{\mathrm{d}\varphi_{\mathbf{q}}}{\mathrm{d}\mathbf{q}}\right|_{\mathbf{q}=\mathbf{0}} \dot{\mathbf{q}} \equiv \mathbf{a}\dot{\mathbf{q}}, \tag{2.7}$$

where $\mathrm{d}\varphi_{\mathbf{q}}/\mathrm{d}\mathbf{q}|_{\mathbf{q}=\mathbf{0}} \equiv \mathbf{a}$, obviously.

Eqs. (2.4) and (2.6) characterize completely the rigid body velocity fields. Assuming the possibility of defining the covariant gradient $\nabla\varphi$ (here ∇ is the gradient with respect to $\mathbf{X}$), in the following J_1 will indicate the set whose typical elements are of the type $(\mathbf{x}, \mathbf{F}, \varphi, \nabla\varphi)$. While $J_1(\mathrm{Vel})$ the set whose typical elements are of the type $(\dot{\mathbf{x}}, \dot{\mathbf{F}}, \dot{\varphi}, \nabla\dot{\varphi})$.

The remarks presented up to this point about configurations may be substantially adapted to every ambient Euclidean space (one, two or three dimensional). Some special choices, which are relevant to the situation studied here, need to be specified.

As declared previously, the apparent reference configuration $\mathscr{B}$ is two dimensional. I assume that $\mathscr{B}$ is divided into N sub-regions $\mathscr{B}_i$ by N *open regular curves* Γ_i defined by functions

$$\mathbf{r}_i : [0, s^*] \to \mathscr{B}, \quad 0 < s_i^* < \infty \tag{2.8}$$

such that

$$|\mathbf{r}_{i,s}| = 1 \quad \forall s \in [0, s_i^*], \tag{2.9}$$

$$\mathbf{r}_{i,s}(\tilde{s}) \neq 0 \quad \forall \tilde{s} \in \left[0, s_i^*\right], \tag{2.10}$$

$$\mathbf{r}_i(s_i^*) = \mathscr{J}, \tag{2.11}$$

$\mathbf{r}_{i,s}$ and $\mathbf{r}_{i,ss}$ are *linearly independent*. Each $\mathscr{B}_i$ is a fit region.

It is worth noting that in each $\mathscr{B}_i$, the nature of the order parameter φ is the same. No mixture among different types of substructures are considered.

In Eq. (2.11), $\mathscr{J}$ represents the junction, which is (in the present case) a point in the plane and belongs (as an *assumption*) to the interior of $\mathscr{B}$.

The tangent field t_i to each Γ_i is given by

$$\ell_i = \mathbf{r}_{i,s}, \tag{2.12}$$

while the curvature vector h_i is given by

$$h_i = \mathbf{r}_{i,ss} \tag{2.13}$$

and the Gaussian curvature k by

$$k_i = h_i \cdot \mathbf{n}_i, \tag{2.14}$$

$\mathbf{n}_i$ being the normal to Γ_i.

In the following, I will consider the curves Γ_i as *discontinuity curves* for some field. Indicated with e arbitrarily, one of these fields, it is assumed $e \in C^n\left(\mathscr{B} \setminus \bigcup_{i=1}^{N} \Gamma_i\right)$, for appropriate [5] n. The limiting values of e at each curve (indicated with e^+ and e^-) are defined by

$$e_i^{\pm} = \lim_{\varepsilon \to 0}(\mathbf{x} \pm \varepsilon \mathbf{n}_i, \ell), \tag{2.15}$$

where ε is a real number.

The *jump* $[e]_i$ at the curve is thus given by

$$[e]_i = e_{(i)}^+ - e_{(i)}^-. \tag{2.16}$$

The following relation holds true: $[e_1 e_2] = [e_1]\langle e_2 \rangle + \langle e_1 \rangle [e_2]$, where $\langle e \rangle = \frac{1}{2}(e^+ + e^-)$.

I assume that all field suffering jumps at Γ_i, for all choices of "i", considered in the sequel of the paper, have *finite jumps* and they may be eventually singular at the junction only.

Moreover, such *discontinuity curves* are considered as *coherent*, i.e. not only

$$[\mathbf{F}]_i(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i) = 0, \tag{2.17}$$

where $\mathbf{I}$ is the *second-order unit tensor*, as in the classical case of coherent discontinuity surfaces in simple materials (Gurtin, 1993a), but also

$$[\nabla\varphi]_i(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i) = 0. \tag{2.18}$$

Of course, Eqs. (2.17) and (2.18) implies $\mathbf{F}_i^+(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i) = \mathbf{F}_i^-(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i)$ and $(\nabla\varphi)_i^+(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i) = (\nabla\varphi)_i^-(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i)$; so, F will indicate in the following indifferently $\mathbf{F}_i^+(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i)$ or $\mathbf{F}_i^-(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i)$ and N indifferently $(\nabla\varphi)_i^+(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i)$ or $(\nabla\varphi)_i^-(\mathbf{I} - \mathbf{n}_i \otimes \mathbf{n}_i)$.

By coming back to e, now, since e can be singular at the junction, some technically clever devices need to be used in applying the Gauss theorem to the divergence of e.

To explain this claim, consider a disc D^* of radius R, centered at $\mathscr{J}$ and included in $\mathscr{B}$, i.e. $D^* \subseteq \mathscr{B}$, and another disc D_r of radius $r < R$ centered at $\mathscr{J}$.

The integral of $\operatorname{Div} e$ on D^* can be defined, when e is a *vector field* [6], as [7]

$$\int_{D^*} \operatorname{Div} e \, \mathrm{dA} \stackrel{\text{def}}{=} \lim_{r \to 0} \int_{D^* \setminus D_r} \operatorname{Div} e \, \mathrm{dA}. \tag{2.19}$$

Now, the Gauss theorem implies

$$\int_{D^* \setminus D_r} \operatorname{Div} e \, \mathrm{dA} = \int_{\partial D^*} e \cdot \mathbf{m} \, \mathrm{d}l - \sum_{i=1}^{\bar{N}} \int_{\Gamma_i \cap (D^* \setminus D_r)} [e]_i \cdot \mathbf{n}_i \, \mathrm{d}l - \int_{\partial D_r} e \cdot \mathbf{m} \, \mathrm{d}l, \tag{2.20}$$

[5] In the following, it will be only necessary to consider n equal to 1 or, at most, to 2.

[6] For higher-order tensor rank fields, Eq. (2.18) should hold for single components.

[7] In the sequel of the paper, integrals on domains containing in their interior the junction must be considered in the sense of Eq. (2.19).

where $\mathbf{m}$ is the normal to $\partial D^* \cup \partial D_r$. Thus, letting r to zero, the following relation holds true:

$$\int_{D^*} \mathrm{Div}\, e \,\mathrm{dA} = \int_{\partial D^*} e \cdot \mathbf{m} \,\mathrm{d}l - \sum_{i=1}^{\bar{N}} \int_{\Gamma_i \cap D^*} [e]_i \cdot \mathbf{n}_i \,\mathrm{d}l - \lim_{r \to 0} \int_{\partial D_r} e \cdot \mathbf{m} \,\mathrm{d}l. \tag{2.21}$$

Moreover, I assume that all fields considered in the sequel of the paper have sufficiently degree of smoothness to assure the existence of the limit, $\lim_{r\to 0}$, of some integrals on D_r or its boundary.

Eq. (2.21) is the basic tool for deducing balance equations from the general expression of the power performed by interactions associated to both placement and order parameter fields.

Note that where the discontinuity lines Γ_i move within $\mathscr{B}$ with normal velocity $\mathscr{U}_i$ (it will be defined in the following) then, by the Hadamard theorem,

$$[\dot{\mathbf{x}}]_{(i)} = -\mathscr{U}_i [\mathbf{F}]_{(i)} \mathbf{n}_i, \tag{2.22}$$

$$[\dot{\varphi}]_{(i)} = -\mathscr{U}_i [\nabla \varphi]_{(i)} \mathbf{n}_i. \tag{2.23}$$

Obviously, when $\mathscr{U}_i = 0$ the velocity fields $\dot{\mathbf{x}}$ and $\dot{\varphi}$ are continuous across the ith line.

3. Balance of standard interactions

As stated by Eq. (2.1), the order parameter φ is considered as an observable quantity. An observer makes two different ideal measures to recognize both the placement and the microstructure of each material patch.

Moreover, interactions should be added to each measurement because they are related to different "kinematical" or "configurational" mechanisms. Such interactions can be evaluated through their contribution to the overall power performed during mechanical processes.

The power $\mathscr{P}$ is a real functional defined on $J_1(\mathrm{Vel})$; in particular, the power performed on a part [8] $\mathscr{B}^*$ of $\mathscr{B}$ is given by a mapping

$$\mathscr{P}_{\mathscr{B}^*} : J_1(\mathrm{Vel}) \to \mathscr{R}, \tag{3.1}$$

which is decomposed into external and inner contributions, $\mathscr{P}^{\mathrm{ext}}$ and $\mathscr{P}^{\mathrm{int}}$, respectively:

$$\mathscr{P} = \mathscr{P}^{\mathrm{ext}} - \mathscr{P}^{\mathrm{int}}. \tag{3.2}$$

Now, a problem is the explicit representation of $\mathscr{P}^{\mathrm{ext}}$ and $\mathscr{P}^{\mathrm{int}}$.

Since $(\dot{\mathbf{x}}, \dot{\mathbf{F}}, \dot{\varphi}, \nabla\dot{\varphi})$ is the typical element of $J_1(\mathrm{Vel})$, it is necessary to consider measures of interaction acting on each component of $(\dot{\mathbf{x}}, \dot{\mathbf{F}}, \dot{\varphi}, \nabla\dot{\varphi})$. In particular, the following measures of interactions are considered:

Interactions on $\mathscr{B}^*$

(a) *External interactions* relating $\mathscr{B}^*$ with the surrounding environment in which is contained including the rest of the body

- $\mathbf{b}$ bulk forces expending power on $\dot{\mathbf{x}}$ (e.g., gravitational forces)
- β bulk "forces" expending power on $\dot{\varphi}$ (electromagnetic fields on microstructures)
- $\mathbf{t}$ boundary traction expending power on $\dot{\mathbf{x}}$ (tension)
- τ boundary generalized "traction" expending power on $\dot{\varphi}$ (generalized tension)

(b) *Inner interactions*

- $\mathbf{s}$ zero stress
- $\mathbf{T}$ Piola–Kirchhoff stress tensor

[8] With the term "part" I indicate a subset of $\mathscr{B}$, which is also a fit region.

z self-forces
$\mathscr{S}$ microstress
Interactions on $\bigcup_{i=1}^{N} \Gamma_i \cap \partial\mathscr{B}^*$
(c) *External interactions*
$\xi_{(i)}$ traction at $\Gamma_i \cap \partial\mathscr{B}^*$
$\zeta_{(i)}$ generalized traction at $\Gamma_i \cap \partial\mathscr{B}^*$
(d) *Inner interactions*
$T_{(i)}$ *line* stress on Γ_i (vector quantity)
$Z_{(i)}$ *line* self-force on Γ_i (tensor quantity of the same rank of φ)
$S_{(i)}$ *line* microstress on Γ_i (tensor quantity of the same rank of φ)

A tetrahedron argument, analogous to the classical one stating that

$$\mathbf{Tm} = \mathbf{t} \quad \text{at } \partial\mathscr{B}^* \tag{3.3}$$

allows us to write

$$\mathscr{S}\mathbf{m} = \tau \quad \text{at } \partial\mathscr{B}^*. \tag{3.4}$$

Moreover, it is necessary that the following equalities hold true at the intersection of each line of discontinuity with the boundary of $\mathscr{B}^*$:

$$T_{(i)} = \xi_{(i)} \quad \text{at } \Gamma_i \cap \partial\mathscr{B}^*, \tag{3.5}$$

$$S_{(i)} = \zeta_{(i)} \quad \text{at } \Gamma_i \cap \partial\mathscr{B}^*. \tag{3.6}$$

Now, consider a part $\mathscr{B}^*$ of $\mathscr{B}$ containing the junction in its interior; $\mathscr{P}^{\text{ext}}$ and $\mathscr{P}^{\text{int}}$ have the following form:

$$\mathscr{P}^{\text{ext}}_{\mathscr{B}^*} = \int_{\mathscr{B}^*} (\mathbf{b} \cdot \dot{\mathbf{x}} + \beta \cdot \dot{\varphi})\,\mathrm{dA} + \int_{\partial\mathscr{B}^*} (\mathbf{t} \cdot \dot{\mathbf{x}} + \tau \cdot \dot{\varphi})\,\mathrm{d}l + \sum_{i=1}^{\bar{N}} \left(\xi_{(i)} \cdot \dot{\mathbf{x}}_i^{\pm} + \xi_{(i)} \cdot \dot{\varphi}_i^{\pm}\right)\Big|_{\partial\mathscr{B}^* \cap \Gamma_i}, \tag{3.7}$$

$$\begin{aligned} \mathscr{P}^{\text{int}}_{\mathscr{B}^*} = {} & \int_{\mathscr{B}^*} (\mathbf{s} \cdot \dot{\mathbf{x}} + \mathbf{T} \cdot \dot{\mathbf{F}} + \mathbf{z} \cdot \dot{\varphi} + \mathscr{S} \cdot \nabla\dot{\varphi})\,\mathrm{dA} + \sum_{i=1}^{\bar{N}} \int_{\mathscr{B}^* \cap \Gamma_i} (T_{(i)} \cdot \dot{F}_i + S_{(i)} \cdot \dot{N}_i + Z_{(i)} \cdot \dot{\varphi}_i^{\pm})\,\mathrm{d}l \\ & + \sum_{i=1}^{\bar{N}} \int_{\mathscr{B}^* \cap \Gamma_i} (\langle \mathbf{T} \rangle_i n_i \cdot [\dot{\mathbf{x}}]_i + \langle \mathscr{S} \rangle_i n_i \cdot [\dot{\varphi}]_i)\,\mathrm{d}l. \end{aligned} \tag{3.8}$$

To derive balance equations from the explicit expression of the power, it is assumed:
(A1) $\mathscr{P}_{\mathscr{B}^*} = 0$, for every choice of $\mathscr{B}^*$ and of the velocity fields $\dot{\mathbf{x}}$ and $\dot{\phi}$;
(A2) $\mathscr{P}^{\text{int}}_{\mathscr{B}^*} = 0$, for every choice of $\mathscr{B}^*$ and of rigid velocity fields.

Some analytical calculations need to be developed to be able to apply A1 to $\mathscr{P}$. Insertion of Eqs. (3.3)–(3.8) into Eq. (3.2) and the use of Gauss theorem lead to

$$\begin{aligned} \mathscr{P}_{\mathscr{B}^*} = {} & \int_{\mathscr{B}^*} ((\mathbf{b} - \mathbf{s} + \mathrm{Div}\,\mathbf{T}) \cdot \dot{\mathbf{x}} + (\beta - z + \mathrm{Div}\,\mathscr{S}) \cdot \dot{\varphi})\mathrm{dA} + \sum_{i=1}^{\bar{N}} \int_{\mathscr{B}^* \cap \Gamma_i} \Big(([\mathbf{T}]_{(i)}\mathbf{n}_i + \partial_s T_{(i)}) \cdot \dot{\mathbf{x}}_i^{\pm} \\ & + ([\mathscr{S}]_{(i)}\mathbf{n}_i - Z_{(i)} + \partial_s S_{(i)}) \cdot \dot{\varphi}_i^{\pm}\Big)\mathrm{d}l + \lim_{r \to 0} \int_{\partial D_r} (\mathbf{Tm} \cdot \dot{\mathbf{x}} + \mathscr{S}\mathbf{m} \cdot \dot{\varphi})\mathrm{d}l \\ & + \lim_{r \to 0} \sum_{i=1}^{\bar{N}} \Big(T_{(i)} \cdot \dot{\mathbf{x}}_i^{\pm} + \mathscr{S}_{(i)} \cdot \dot{\varphi}_i^{\pm}\Big)\Big|_{\left(\left(\mathscr{B}^* \setminus \mathring{D}_r\right) \cap \partial D_r\right) \cap \Gamma_i}. \end{aligned} \tag{3.9}$$

Of course, in Eq. (3.9), the disc D_r has been chosen as in Section 2 to be fully contained in $\mathscr{B}^*$ in a way such that $D_r \cap \partial\mathscr{B}^* = \emptyset$.

Note that, since the velocity fields $\dot{\mathbf{x}}$ and $\dot{\varphi}$ are considered to be *not singular* at the junction, the following relations hold true:

$$\lim_{r\to 0}\int_{\partial D_r}(\mathbf{Tm}\cdot\dot{\mathbf{x}}+\mathscr{S}\mathbf{m}\cdot\dot{\varphi})\,\mathrm{d}l=\left(\lim_{r\to 0}\int_{\partial D_r}\mathbf{Tm}\,\mathrm{d}l\right)\cdot\dot{\mathbf{x}}_{\mathscr{J}}+\left(\lim_{r\to 0}\int_{\partial D_r}\mathscr{S}\mathbf{m}\,\mathrm{d}l\right)\cdot\dot{\varphi}_{\mathscr{J}}, \tag{3.10}$$

$$\begin{aligned}\lim_{r\to 0}\sum_{i=1}^{N}\left(T_{(i)}\cdot\dot{\mathbf{x}}_i^{\pm}+S_{(i)}\cdot\dot{\varphi}_{\mathrm{i}}^{\pm}\right)\Bigg|_{\left(\left(\mathscr{B}^*\setminus\mathring{D}_r\right)\cap\partial D_r\right)\cap\Gamma_i}&=\left(\lim_{r\to 0}\sum_{i=1}^{N}T_{(i)}\Bigg|_{((\mathscr{B}^*\setminus\mathring{D}_r)\cap\partial D_r)\cap\Gamma_{i_i}}\right)\cdot\dot{\mathbf{x}}_{\mathscr{J}}\\&\quad+\left(\lim_{r\to 0}\sum_{i=1}^{N}S_{(i)}\Bigg|_{((\mathscr{B}^*\mathring{D}_r)\cap\partial D_r)\cap\Gamma_{i_i}}\right)\cdot\dot{\varphi}_{\mathscr{J}},\end{aligned} \tag{3.11}$$

where $\dot{\mathbf{x}}_{\mathscr{J}}$ is the velocity field calculated at the junction and $\dot{\varphi}_{\mathscr{J}}$ is the time rate of the order parameter at the junction itself.

Thus, taking into account Eqs. (3.10) and (3.11), A1 implies the following balance equations:

$$\mathbf{b}-\mathbf{s}+\mathrm{Div}\,\mathbf{T}=0\quad\text{in }\mathscr{B}, \tag{3.12}$$

$$\beta-\mathbf{z}+\mathrm{Div}\,\mathscr{S}=0\quad\text{in }\mathscr{B}, \tag{3.13}$$

$$[\mathbf{T}]_{(i)}\mathbf{n}_i+\partial_{s}T_{(i)}=0\quad\text{on }\Gamma_i, \tag{3.14}$$

$$[\mathscr{S}]_{(i)}\mathbf{n}_i-Z_{(i)}+\partial_{s}S_{(i)}=0\quad\text{on }\Gamma_i, \tag{3.15}$$

$$\lim_{r\to 0}\left(\int_{\partial D_r}\mathbf{Tm}\,\mathrm{d}l+\sum_{i=1}^{N}T_{(i)}\Bigg|_{\left(\left(\mathscr{B}^*\setminus\mathring{D}_r\right)\cap\partial D_r\right)\cap\Gamma_{\mathrm{i}}}\right)=0\qquad\text{at }\mathscr{J}, \tag{3.16}$$

$$\lim_{r\to 0}\left(\int_{\partial D_r}\mathscr{S}\mathbf{m}\,\mathrm{d}l+\sum_{i=1}^{N}S_{(i)}\Bigg|_{\left(\left(\mathscr{B}^*\setminus\mathring{D}_r\right)\cap\partial D_r\right)\cap\Gamma_i}\right)=0\qquad\text{at }\mathscr{J}. \tag{3.17}$$

The consequences of A2 can be deduced taking into account that

$$\int_{\mathscr{B}^*}(\mathbf{s}\cdot\dot{\mathbf{x}}+\mathbf{T}\cdot\dot{\mathbf{F}}+\mathbf{z}\cdot\dot{\varphi}+\mathscr{S}\cdot\nabla\dot{\varphi})\,\mathrm{dA} \tag{3.18}$$

should vanish, for every choice of $\mathscr{B}^*$, and for every choice of rotational velocity $\dot{\mathbf{q}}$, where $\dot{\mathbf{x}}$ and $\dot{\varphi}$ are chosen according to Eqs. (2.5) and (2.7).

The substitution of Eqs. (2.5) and (2.6) into Eq. (3.18) and the application of A2 lead to [9]

$$\mathbf{s}=0, \tag{3.19}$$

[9] In Eq. (3.20), the superscript "t" means transposition defined on tensors of rank greater than or equal to 2 as follows: $\mathbf{A}^{\mathrm{t}}\mathbf{v}=\mathbf{v}\mathbf{A}$.
For every vector $\mathbf{v}$ and tensor $\mathbf{A}$. Of course, $\mathbf{A}^{\mathrm{t}}$ coincides with $\mathbf{A}^{\mathrm{T}}$ when $\mathbf{A}$ is a second-order tensor.

$$\mathbf{eTF}^{\mathrm{T}} = \mathbf{az} + (\nabla\mathbf{a})^{\mathrm{t}}\mathscr{S}. \tag{3.20}$$

Of course, if T and S are not singular at the junction (as will be assumed in the following) Eqs. (3.16) and (3.17) reduce to

$$\lim_{r\to 0}\int_{\partial D_r}\mathbf{Tm}\,\mathrm{d}l + \sum_{i=1}^{N} T_{(i)}\Big|_{\mathscr{J}} = 0 \quad \text{at } \mathscr{J}, \tag{3.21}$$

$$\lim_{r\to 0}\int_{\partial D_r}\mathscr{S}\mathbf{m}\,\mathrm{d}l + \sum_{i=1}^{\bar{N}} S_{(i)}\Big|_{\mathscr{J}} = 0 \quad \text{at } \mathscr{J}. \tag{3.22}$$

4. Balance of configurational interactions

The discontinuity lines move within the body when phenomena like phase transitions occur. In this case, the representation function of each line is of the type $\mathbf{r}(s,t)$, i.e. it depends on time, so that the velocity of the line in the plane is given by

$$\mathbf{u} = \partial_t\mathbf{r}. \tag{4.1}$$

Note that only the normal component of $\mathbf{u}$ to Γ

$$\mathscr{U} = \mathbf{u}\cdot\mathbf{n} \tag{4.2}$$

is intrinsic. [10]

The movement of the lines Γ implies the need of considering $\mathscr{B}^*$ as varying under time; so $\mathscr{B}^* = \mathscr{B}^*(t)$. Moreover, $\mathscr{B}^*$ can be also chosen in a way such that its boundary $\partial\mathscr{B}^*$ can be parameterized by arc length u. If $\mathbf{X} = \hat{\mathbf{X}}(u,t)$ represents $\partial\mathscr{B}^*(t)$,

$$\mathbf{v} = \partial_t\hat{\mathbf{X}}(u,t) \tag{4.3}$$

is the velocity of the boundary $\partial\mathscr{B}^*$ whose normal component is given by

$$\mathscr{V} = \mathbf{v}\cdot\mathbf{n}. \tag{4.4}$$

When $\mathscr{B}^*$ is deformed, the velocity of the boundary is given by

$$\bar{\mathbf{v}} = \overset{\bullet}{\overline{\hat{\mathbf{x}}(\mathbf{X}(u,t),t)}} = \dot{\mathbf{x}} + \mathbf{F}\mathbf{v}. \tag{4.5}$$

An analogous relation holds true for the time rate of the order parameter

$$D_t\varphi = \overset{\bullet}{\overline{\hat{\varphi}(\mathbf{X}(u,t),t)}} = \dot{\varphi} + (\nabla\varphi)\mathbf{v}. \tag{4.6}$$

From analogous calculations, the current velocities at the interfaces Γ are given by

$$\bar{\mathbf{u}} = \dot{\mathbf{x}}^{\pm} + \mathbf{F}^{\pm}\mathbf{u} = \langle\dot{\mathbf{x}}\rangle + \langle\mathbf{F}\rangle\mathbf{u}, \tag{4.7}$$

$$\tilde{D}_t\varphi = \dot{\varphi}^{\pm} + (\nabla\varphi)^{\pm}\mathbf{u} = \langle\dot{\mathbf{x}}\rangle + \langle\nabla\varphi\rangle\mathbf{u}. \tag{4.8}$$

[10] In Eq. (4.2) the subscript "i" is omitted because Eq. (4.2) is referred to each Γ_i.

The kinematics of discontinuity lines (including the junction) just described is a mechanism independent of the overall behavior. Such a relative motion with respect to the rest of the body implies interactions because the rest of the body can obstruct or drive the interface and the junction. In addition, the junction is assumed to be *only one point during the whole evolution*.

The interactions arising during the evolution owing to the change under time of a part of the reference configuration are also called configurational forces (Nabarro, 1985; Ericksen, 1995).

I will consider the following configurational measures of interaction:

(a) *Bulk configurational forces*

$\mathbf{P}$ modified Eshelby tensor

$\mathcal{g}$ inner configurational body force

$\mathcal{e}$ external configurational body force

(b) *Surface configurational forces*

$\boldsymbol{c}_i$ line configurational stress at Γ_i

f_i inner line configurational force at Γ_i

(c) *Configurational force at the junction*

$\boldsymbol{d}_{\mathcal{J}}$

I assume that

- $\mathbf{P}$ has finite jump at each Γ_i and can be singular at $\mathcal{J}$,
- $\mathcal{g}$ and $\mathcal{e}$ are such that

$$\int_{\mathscr{B}^*(\ell)} (\mathcal{g} + \mathcal{e}) \mathrm{dA} = \lim_{r \to 0} \int_{\mathscr{B}^*(\ell) \setminus D_r(\ell)} (\mathcal{g} + \mathcal{e}) \mathrm{dA} \quad \forall \ell. \tag{4.9}$$

The same property holds true for f, so

$$\int_{\Gamma_i(\ell) \cap \mathscr{B}^*(\ell)} \mathsf{f}_i \, \mathrm{d}l = \lim_{r \to 0} \int_{\Gamma_i(\ell) \cap (\mathscr{B}^*(\ell) \setminus D_r(\ell))} \mathsf{f}_i \mathrm{d}l \quad \forall \ell. \tag{4.10}$$

- $\boldsymbol{c}$ is not singular at $\mathcal{J}$.

Since $\mathcal{g}$ and f are inner forces associated to the independent kinematics of the interfaces, they do not perform explicit power except in the case in which the power itself is evaluated by an *external observer* not fixed, rather *migrating with constant velocity* $\mathbf{v}^*$ and evaluating the position of the moving reference control volume $\mathscr{B}^*(\ell)$. The same situation occurs for e.

Indicating by $W^*(\mathscr{B}^*)$, the power performed on $\mathscr{B}^*(\ell)$ by all interactions with respect to the migrating observer just introduced, $W^*(\mathscr{B}^*)$ can be expressed as

$$\begin{aligned} W^*(\mathscr{B}^*) = & \int_{\partial \mathscr{B}^*(\ell)} (\mathbf{Tm} \cdot \bar{\mathbf{v}} + \mathscr{S}\mathbf{m} \cdot D_\ell \varphi + \mathbf{Pm} \cdot (\mathbf{v} + \mathbf{v}^*)) \mathrm{d}l + \int_{\mathscr{B}^*(\ell)} (\mathbf{b} \cdot \dot{\mathbf{x}} + \beta \cdot \dot{\varphi}) \mathrm{dA} \\ & + \int_{\mathscr{B}^*(\ell)} \left(\mathcal{g} + \mathcal{e} \right) \cdot \mathbf{v}^* \mathrm{dA} + \sum_{i=1}^{\bar{N}} (T_{(i)} \cdot \bar{\mathbf{u}}_i + S_{(i)} \cdot \tilde{D}_\ell \varphi) \Big|_{\Gamma_i(\ell) \cap \partial \mathscr{B}^*(\ell)} + \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell) \cap \mathscr{B}^*(\ell)} \mathsf{f}_{(i)} \cdot \mathbf{v}^* \mathrm{d}l \\ & + \sum_{i=1}^{\bar{N}} \left(\boldsymbol{c}_{(i)} \cdot (\mathbf{u} + \mathbf{v}^*) \right) \Big|_{\Gamma_i(\ell) \cap \partial \mathscr{B}^*(\ell)} + \boldsymbol{d}_{\mathcal{J}} \cdot \mathbf{v}^*. \end{aligned} \tag{4.11}$$

$W^*(\mathscr{B}^*)$ (where Eqs. (3.3)–(3.6) have been considered) accounts for the power performed by $\mathcal{g}$, $\mathcal{e}$ and $\boldsymbol{d}_{\mathcal{J}}$ on the velocity $\mathbf{v}^*$ of the material observer evaluating $\mathscr{B}^*(\ell)$.

Of course, to be physically consistent, $W^*(\mathscr{B}^*)$ *should be independent of the choice of the velocity of the material observer*. [11] Such a requirement implies that

$$\left(\int_{\partial \mathscr{B}^*(\ell)} \mathbf{Pm}\, \mathrm{d}l + \int_{\mathscr{B}^*(\ell)} (\mathscr{g} + e) \mathrm{dA} + \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \mathsf{f}_{(i)}\, \mathrm{d}l + \sum_{i=1}^{\bar{N}} \boldsymbol{c}_{(i)} \Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} + \boldsymbol{d}_{\mathscr{J}} \right) \cdot \mathbf{v}^* = 0. \tag{4.12}$$

From Eq. (4.12), since $\mathbf{v}^*$ is arbitrary, the *integral configurational balance* follows:

$$\int_{\partial \mathscr{B}^*(\ell)} \mathbf{Pm}\, \mathrm{d}l + \int_{\mathscr{B}^*(\ell)} (\mathscr{g} + e) \mathrm{dA} + \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \mathsf{f}_{(i)}\, \mathrm{d}l + \sum_{i=1}^{\bar{N}} \boldsymbol{c}_{(i)} \Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} + \boldsymbol{d}_{\mathscr{J}} = 0. \tag{4.13}$$

Now, from the Gauss theorem,

$$\int_{\partial \mathscr{B}^*(\ell)} \mathbf{Pm}\, \mathrm{d}l = \int_{\mathscr{B}^*(\ell)} \mathrm{Div}\, \mathbf{P}\, \mathrm{dA} + \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} [\mathbf{P}]_{(i)} \mathbf{n}_i\, \mathrm{d}l + \lim_{r\to 0} \int_{\partial D_r(\ell)} \mathbf{Pm}\, \mathrm{d}l, \tag{4.14}$$

$$\begin{aligned} \sum_{i=1}^{\bar{N}} \boldsymbol{c}_{(i)} \Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} &= \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \partial_s \boldsymbol{c}_{(i)}\, \mathrm{d}l + \lim_{r\to 0} \sum_{i=1}^{\bar{N}} \boldsymbol{c}_{(i)} \Big|_{\Gamma_i(\ell)\cap D_r(\ell)} \\ &= \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \partial_s \boldsymbol{c}_{(i)}\, \mathrm{d}l + \sum_{i=1}^{\bar{N}} \boldsymbol{c}_{(i)} \Big|_{\mathscr{J}}. \end{aligned} \tag{4.15}$$

Eq. (4.15) is a consequence of the previous assumption that $\boldsymbol{c}$ is not singular at the junction.

Since Eq. (4.13) should be valid for every choice of the control volume $\mathscr{B}^*$, insertion of Eqs. (4.14) and (4.15) in Eq. (4.13) leads to the following localized balance of configurational forces, namely

$$\mathrm{Div}\, \mathbf{P} + \mathscr{g} + e = 0 \quad \text{in } \mathscr{B}, \tag{4.16}$$

$$[\mathbf{P}]_{(i)} \mathbf{n}_i + \mathsf{f}_i + \partial_s \boldsymbol{c}_{(i)} = 0 \quad \text{on } \Gamma_i, \tag{4.17}$$

$$\lim_{r\to 0} \int_{\partial D_r(\ell)} \mathbf{Pm}\, \mathrm{d}l + \sum_{i=1}^{\bar{N}} \boldsymbol{c}_{(i)} \Big|_{\mathscr{J}} + \boldsymbol{d}_{\mathscr{J}} = 0 \quad \text{at } \mathscr{J}. \tag{4.18}$$

5. Configurational force at the two-dimensional junction

Now, the determination of the explicit expressions of configurational interactions needs to be made. Such expressions can be done in terms of standard bulk and surface stresses, free energies and geometrical ingredients like the deformation gradient and the gradient of the order parameter.

The result follows from the exploitation of an isothermal expression of the second law of thermodynamics stating that

[11] The requirements of arbitrary choices of the velocity fields are considered here to be valid in the absence of relativistic effects, i.e. when problems related to the existence of a limit speed of transfer of information can be neglected.

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\{\text{free energy in } \mathscr{B}^*(\ell)\} - W(\mathscr{B}^*) \leqslant 0. \tag{5.1}$$

Here, $W(\mathscr{B}^*)$ is the power expressed with reference to a fixed material observer ($\mathbf{v}^* = 0$); so, it has the following form:

$$W(\mathscr{B}^*) = \int_{\partial\mathscr{B}^*(\ell)} (\mathbf{Tm}\cdot\bar{\mathbf{v}} + \mathscr{S}\mathbf{m}\cdot D_\ell\varphi + \mathbf{Pm}\cdot\mathbf{v})\mathrm{d}l + \int_{\mathscr{B}^*(\ell)} (\mathbf{b}\cdot\dot{\mathbf{x}} + \beta\cdot\dot{\varphi})\mathrm{dA} + \sum_{i=1}^{\bar{N}} (T_{(i)}\cdot\bar{\mathbf{u}}_i + S_{(i)}\cdot\tilde{D}_\ell\varphi_i + \boldsymbol{c}_{(i)}\cdot\mathbf{u}_i)\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)}. \tag{5.2}$$

On the basis of the experimental physical evidence, a *bulk free energy density* ψ and a *line free energy* ϕ should be considered. So that, the time rate of the free energy has the following explicit expression:

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\{\text{free energy in } \mathscr{B}^*(\ell)\} = \frac{\mathrm{d}}{\mathrm{d}\ell}\int_{\mathscr{B}^*(\ell)} \psi\,\mathrm{dA} + \frac{\mathrm{d}}{\mathrm{d}\ell}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \phi\,\mathrm{dA}. \tag{5.3}$$

In particular, I assume that
- ψ is smooth on $\mathscr{B}\setminus\bigcup_{i=1}^{N}\Gamma_i$, suffers finite jump at each Γ_i and could be singular at the junction;
- ϕ is smooth on $\left(\bigcup_{i=1}^{N}\Gamma_i\right)$.

Now, the standard transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\int_{\mathscr{B}^*(\ell)\setminus D_r(\ell)} \psi\,\mathrm{dA} = \int_{\mathscr{B}^*(\ell)\setminus D_r(\ell)} \dot{\psi}\,\mathrm{dA} + \int_{\partial\mathscr{B}^*(\ell)} \psi\mathscr{V}\,\mathrm{d}l - \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap(\mathscr{B}^*(\ell)\setminus D_r(\ell))} [\psi]_{(i)}\mathscr{U}_i\,\mathrm{d}l - \int_{\partial D_r(\ell)} \psi\mathscr{V}\,\mathrm{d}l. \tag{5.4}$$

Taking the limit $r \to 0$ and considering previous assumptions, the following equation holds true:

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\int_{\mathscr{B}^*(\ell)} \psi\,\mathrm{dA} = \int_{\mathscr{B}^*(\ell)} \dot{\psi}\,\mathrm{dA} + \int_{\partial\mathscr{B}^*(\ell)} \psi\mathscr{V}\,\mathrm{d}l - \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} [\psi]_{(i)}\mathscr{U}_i\,\mathrm{d}l - \lim_{r\to 0}\int_{\partial D_r(\ell)} \psi\mathscr{V}\,\mathrm{d}l. \tag{5.5}$$

Analogous calculations on the basis of standard transport theorems for integrals calculated on lines evolving under time furnish

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \phi\,\mathrm{d}l = \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\phi^0 - \phi\ell\mathscr{U})\mathrm{d}l + \phi(\mathbf{u}_{(i)}\cdot t_i)\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} - \phi(\mathbf{u}_{(i)}\cdot t_i)\Big|_{\mathscr{J}}. \tag{5.6}$$

In Eq. (5.6) ϕ^0 represents the normal time derivative of ϕ, following the trajectories. [12] To simplify calculations, we now set

$$W_1(\mathscr{B}^*) = \int_{\partial\mathscr{B}^*(\ell)} (\mathbf{Tm}\cdot\bar{\mathbf{v}} + \mathscr{S}\mathbf{m}\cdot D_\ell\varphi + \mathbf{Pm}\cdot\mathbf{v})\mathrm{d}l, \tag{5.7}$$

$$W_2(\mathscr{B}^*) = \sum_{i=1}^{\bar{N}} (T_{(i)}\cdot\bar{\mathbf{u}}_i + S_{(i)}\cdot\tilde{D}_\ell\varphi_i + \boldsymbol{c}_{(i)}\cdot\mathbf{u}_i)\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)}. \tag{5.8}$$

The insertion of Eqs. (4.5) and (4.6) in Eq. (5.7) and of Eqs. (4.7) and (4.8) in Eq. (5.8) leads, respectively, to

[12] More precisely, ϕ^0 is defined as follows: $\phi^0(\mathbf{x},\mathrm{t}) = \mathrm{d}\phi(\mathbf{x}(\beta),\beta)/\mathrm{d}\beta\big|_{\beta=\ell}$, where $\mathbf{x}(\beta)$ satisfies $\mathrm{d}\mathbf{x}/\mathrm{d}\beta = \mathscr{U}(\mathbf{x}(\beta),\beta)$.

$$W_1(\mathscr{B}^*) = \int_{\partial\mathscr{B}^*(\ell)} (\mathbf{Tm}\cdot\dot{\mathbf{x}} + \mathscr{S}\mathbf{m}\cdot\dot{\varphi} + \left(\mathbf{P} + \mathbf{F}^{\mathrm{T}}\mathbf{T} + \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right)\mathbf{m}\cdot\mathbf{v})\mathrm{d}l, \tag{5.9}$$

$$W_2(\mathscr{B}^*) = \sum_{i=1}^{\bar{N}} (T_{(i)}\cdot\langle\dot{\mathbf{x}}\rangle_i + S_{(i)}\cdot\langle\varphi\rangle_i + (\boldsymbol{c}_{(i)} + \langle\mathbf{F}^{\mathrm{T}}\rangle_i T_{(i)} + \langle\nabla\varphi^{\mathrm{t}}\rangle_i \overset{\bullet}{*} S_{(i)})\cdot\mathbf{u})\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)}. \tag{5.10}$$

For physical reasons, $W_1(\mathscr{B}^*)$ *must be independent of the kind of parameterization of the boundary* of $\mathscr{B}^*(\ell)$. In Eq. (5.9), the term depending of the parameterization is the product $(\mathbf{P} + \mathbf{F}^{\mathrm{T}}\mathbf{T} + \nabla\varphi^{\mathrm{t}} * \mathscr{S})\mathbf{m}\cdot\mathbf{v}$ because only the normal component of $\mathbf{v}$ is intrinsic. As a consequence, it is necessary that

$$\left(\mathbf{P} + \mathbf{F}^{\mathrm{T}}\mathbf{T} + \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right)\mathbf{m}\cdot\mathrm{i} = 0 \tag{5.11}$$

for every vector i chosen to be tangent to $\partial\mathscr{B}^*$, i.e. $(\mathbf{P} + \mathbf{F}^{\mathrm{T}}\mathbf{T} + \nabla\varphi^{\mathrm{t}} * \mathscr{S})\mathbf{m}$ is purely normal; thus, it must be expressed by

$$\mathbf{P} + \mathbf{F}^{\mathrm{T}}\mathbf{T} + \nabla\varphi^{\mathrm{t}} * \mathscr{S} = \omega\mathbf{I}, \tag{5.12}$$

ω being a scalar quantity and $\mathbf{I}$, the unit second-order tensor. Thus, $W_1(\mathscr{B}^*)$ reduces to

$$W_1(\mathscr{B}^*) = \int_{\partial\mathscr{B}^*(\ell)} (\mathbf{Tm}\cdot\dot{\mathbf{x}} + \mathscr{S}\mathbf{m}\cdot\dot{\varphi})\mathrm{d}l + \int_{\partial\mathscr{B}^*(\ell)} \omega\mathscr{V}\,\mathrm{d}l. \tag{5.13}$$

Moreover, in Eq. (5.10), the term $\boldsymbol{c}_{(i)} + \langle\mathbf{F}^{\mathrm{T}}\rangle_i T_{(i)} + \langle\nabla\varphi^{\mathrm{t}}\rangle_i \overset{\bullet}{*} S_{(i)}$ is a vector quantity and can be decomposed in its tangential and normal components to each Γ_i. Thus, omitting the index "i" (in this way, understanding that the calculations developed hold for each Γ_i, "i" fixed), it follows that

$$\boldsymbol{c} + \langle\mathbf{F}^{\mathrm{T}}\rangle T + \langle\nabla\varphi^{\mathrm{t}}\rangle \overset{\bullet}{*} S = \sigma t + \mu\mathbf{n}. \tag{5.14}$$

Now, since

$$\langle\mathbf{F}^{\mathrm{T}}\rangle T = \boldsymbol{F}^{\mathrm{T}}T + (\langle\mathbf{F}\rangle\mathbf{n}\cdot T)\mathbf{n}, \tag{5.15}$$

$$\langle\nabla\varphi^{\mathrm{t}}\rangle \overset{\bullet}{*} S = N^{\mathrm{t}} \overset{\bullet}{*} S + (\langle\nabla\varphi\rangle\mathbf{n}\cdot S)\mathbf{n}, \tag{5.16}$$

the configurational stress vector $\boldsymbol{c}$ may be expressed by

$$\boldsymbol{c} = \sigma t + \bar{\mu}\mathbf{n} - \boldsymbol{F}^{\mathrm{T}}T - N^{\mathrm{t}} \overset{\bullet}{*} S, \tag{5.17}$$

where

$$\bar{\mu} = \mu - \langle\mathbf{F}\rangle\mathbf{n}\cdot T - \langle\nabla\varphi\rangle\mathbf{n}\cdot S. \tag{5.18}$$

Thus, inserting Eq. (5.17) in Eq. (5.10) and once more using Eqs. (5.15) and (5.16), $W_2(\mathscr{B}^*)$ can be written as

$$W_2(\mathscr{B}^*) = \sum_{i=1}^{\bar{N}} (\sigma_{(i)}\left(t_{(i)}\cdot\mathbf{u}_i\right) + \bar{\mu}_{(i)}\mathscr{U}_i + T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle_i + \mathscr{U}_i\langle\mathbf{F}\rangle_{(i)}\mathbf{n}_{(i)}) + S_{(i)}\cdot(\langle\dot{\varphi}\rangle_i + \mathscr{U}_i\langle\nabla\varphi\rangle_{(i)}\mathbf{n}_{(i)}))\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)}. \tag{5.19}$$

On each discontinuity line Γ_i, by applying the Gauss theorem,

$$
\begin{aligned}
&(\bar{\mu}_{(i)}\mathscr{U}_i + T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle_i + \mathscr{U}_i\langle\mathbf{F}\rangle_{(i)}\mathbf{n}_{(i)}) + S_{(i)}\cdot(\langle\dot{\varphi}\rangle_i + \mathscr{U}_i\langle\nabla\varphi\rangle_{(i)}\mathbf{n}_{(i)}))\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} \\
&\quad = \int_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} \partial_s(\bar{\mu}_{(i)}\mathscr{U}_i + T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle + \mathscr{U}\langle\mathbf{F}\rangle\mathbf{n})_i + S_{(i)}\cdot(\langle\dot{\varphi}\rangle + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})_i)\,\mathrm{d}l + (\bar{\mu}_{(i)}\mathscr{U}_i + T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle \\
&\qquad + \mathscr{U}\langle\mathbf{F}\rangle\mathbf{n})_i + S_{(i)}\cdot(\langle\dot{\varphi}\rangle + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})_i)\Big|_{\mathscr{J}}.
\end{aligned} \tag{5.20}
$$

Moreover, since $\bar{\mu} = \boldsymbol{c}\cdot\mathbf{n}$,

$$
\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(t)} \partial_s(\bar{\mu}_{(i)}\mathscr{U}_i)\,\mathrm{d}l = \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \mathscr{U}(\partial_s(\boldsymbol{c}_{(i)})\cdot\mathbf{n}_{(i)} + \boldsymbol{c}_{(i)}\cdot\partial_s\mathbf{n}_{(i)})\,\mathrm{d}l + \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\bar{\mu}_{(i)}\partial_s\mathscr{U}_i)\,\mathrm{d}l, \tag{5.21}
$$

$$
\begin{aligned}
\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \partial_s(T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle + \mathscr{U}\langle\mathbf{F}\rangle\mathbf{n})_i)\,\mathrm{d}l = \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (&\partial_s T_{(i)}\cdot\langle\dot{\mathbf{x}}\rangle_i + \partial_s T_{(i)}\cdot\mathscr{U}_i\langle\mathbf{F}\rangle_{(i)}\mathbf{n}_{(i)} + T_{(i)}\cdot\mathbf{p}^0_{(i)} \\
&- k_{(i)}\mathbf{F}^{\mathrm{T}}_{(i)}T_{(i)}\cdot t_{(i)})\,\mathrm{d}l,
\end{aligned} \tag{5.22}
$$

$$
\begin{aligned}
\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \partial_s(S_{(i)}\cdot(\langle\dot{\varphi}\rangle + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})_i)\,\mathrm{d}l = \int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (&\partial_s S_{(i)}\cdot\langle\dot{\varphi}\rangle_i + \partial_s S_{(i)}\cdot\mathscr{U}_i\langle\nabla\varphi\rangle_{(i)}\mathbf{n}_{(i)} + S_{(i)}\cdot\mathbf{q}^0_{(i)} \\
&- k_{(i)}N^{\mathrm{t}}_{(i)}S_{(i)}\cdot t_{(i)})\,\mathrm{d}l,
\end{aligned} \tag{5.23}
$$

where

$$
\mathbf{p}^0_{(i)} = \partial_s(\langle\dot{\mathbf{x}}\rangle + \mathscr{U}\langle\mathbf{F}\rangle\mathbf{n})_i + k_{(i)}\boldsymbol{F}^{\mathrm{T}}_{(i)}T_{(i)}\cdot t_{(i)}, \tag{5.24}
$$

$$
\mathbf{q}^0_{(i)} = \partial_s(\langle\dot{\varphi}\rangle + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})_i + k_{(i)}N^{\mathrm{t}}_{(i)}\overset{\bullet}{*}S_{(i)}\cdot t_{(i)}. \tag{5.25}
$$

Of course, from Eqs. (5.24) and (5.25), it follows that $\mathbf{p}_i = \partial_s\mathbf{x}_i^{\pm}$ and $\mathbf{q}_i = \partial_s\varphi_i^{\pm}$ at each discontinuity line.

Thus, by taking account of Eqs. (5.13), (5.19) and Eqs. (5.20)–(5.23), and also using the Gauss theorem on the first integral of Eq. (5.13), $W(\mathscr{B}^*)$ can be written as

$$
\begin{aligned}
W(\mathscr{B}^*) &= \int_{\partial\mathscr{B}^*(\ell)} \omega\mathscr{V}\,\mathrm{d}l + \int_{\mathscr{B}^*(\ell)} (\mathbf{T}\cdot\dot{\mathbf{F}} + \mathbf{z}\cdot\dot{\varphi} + \mathscr{S}\cdot\nabla\dot{\varphi})\mathrm{d}v + \lim_{r\to 0}\int_{\partial D_r} \mathbf{Tm}\cdot\dot{\mathbf{x}}\,\mathrm{d}l \\
&\quad + \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} \bar{\mu}\partial_s\mathscr{U}_i\,\mathrm{d}l + \lim_{r\to 0}\int_{\partial D_r} \mathscr{S}\mathbf{m}\cdot\varphi\,\mathrm{d}l + \sum_{i=1}^{\bar{N}}\Big(\sigma_i(t\cdot\mathbf{u})|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)}\Big) \\
&\quad + \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} Z\cdot(\langle\dot{\varphi}\rangle + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})_i\,\mathrm{d}l + \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\partial_s\boldsymbol{c})\cdot\mathbf{n}\mathscr{U}_i\,\mathrm{d}l \\
&\quad + \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\boldsymbol{c}\cdot\partial_s\mathbf{n})_i\mathscr{U}_i\,\mathrm{d}l + \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(t)} T\cdot\mathbf{p}^0\,\mathrm{d}l - \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} k_i\mathscr{U}_i\mathbf{F}^{\mathrm{T}}T\cdot t\,\mathrm{d}l \\
&\quad - \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} [\mathbf{Tn}\cdot\mathbf{Fn}]_{(i)}\mathscr{U}_i\,\mathrm{d}l + \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} S\cdot\mathbf{q}^0\,\mathrm{d}l \\
&\quad - \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} k_i\mathscr{U}_iN^{\mathrm{t}}\overset{\bullet}{*}S\cdot t\,\mathrm{d}l - \sum_{i=1}^{\bar{N}}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} [\mathscr{S}\mathbf{n}\cdot(\nabla\varphi)\mathbf{n}]_i\mathscr{U}_i\,\mathrm{d}l + \sum_{i=1}^{\bar{N}}(\bar{\mu}_{(i)}\mathscr{U}_i \\
&\quad + T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle_i + \mathscr{U}_i\langle\mathbf{F}\rangle_{(i)}\mathbf{n}_{(i)}) + S_{(i)}\cdot(\langle\dot{\varphi}\rangle_i + \mathscr{U}_i\langle\nabla\varphi\rangle_{(i)}\mathbf{n}_{(i)}))|_{\mathscr{J}}.
\end{aligned} \tag{5.26}
$$

In this way, the dissipation inequality becomes

$$\begin{aligned}
&\int_{\mathscr{B}^*(\ell)} (\dot{\psi} - \mathbf{T}\cdot\dot{\mathbf{F}} - \mathbf{z}\cdot\dot{\varphi} - \mathscr{S}\cdot\nabla\dot{\varphi})\mathrm{d}v + \int_{\partial\mathscr{B}^*(\ell)} (\psi-\omega)\mathscr{V}\,\mathrm{d}l - \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} Z\cdot(\langle\dot{\varphi}\rangle \\
&\quad + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})\,\mathrm{d}l + \sum_{i=1}^{\bar N}(\phi_i-\sigma_i)(t\cdot\mathbf{u})\Big|_{\Gamma_i(\ell)\cap\partial\mathscr{B}^*(\ell)} - \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\partial_{\mathscr{s}}\boldsymbol{c})_i\cdot\mathbf{n}_i\mathscr{U}_i\,\mathrm{d}l \\
&\quad - \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\boldsymbol{c}_i\cdot\partial_{\mathscr{s}}\mathbf{n})_i\mathscr{U}_i\,\mathrm{d}l - \lim_{r\to 0}\int_{\partial D_r}(\psi\mathscr{V} + \mathbf{Tm}\cdot\dot{\mathbf{x}} + \mathscr{S}\mathbf{m}\cdot\dot{\varphi})\,\mathrm{d}l + \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\phi_i^0 \\
&\quad - T\cdot\mathbf{p}^0 - S\cdot\mathbf{q}^0 - \bar{\mu}_{(i)}\partial_{\mathscr{s}}\mathscr{U}_i)\,\mathrm{d}l - \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\phi - \boldsymbol{F}^{\mathrm{T}}T\cdot t - N^{\mathrm{t}}\overset{\bullet}{*}S\cdot t)_{(i)}\mathscr{k}_i\mathscr{U}_i\,\mathrm{d}l \\
&\quad - \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} [\psi - \mathbf{Tn}\cdot\mathbf{Fn} - \mathscr{S}\mathbf{n}\cdot(\nabla\varphi)\mathbf{n}]_{(i)}\mathscr{U}_i\,\mathrm{d}l - \sum_{i=1}^{\bar N}(\phi_i(t_i\cdot\mathscr{U}_i) + \bar{\mu}_{(i)}\mathscr{U}_i + T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle \\
&\quad + \mathscr{U}\langle\mathbf{F}\rangle\mathbf{n})_i + S_{(i)}\cdot(\langle\dot{\varphi}\rangle + \mathscr{U}\langle\nabla\varphi\rangle\mathbf{n})_i)\Big|_{\mathscr{J}} \leqslant 0.
\end{aligned}\tag{5.27}$$

Since Eq. (5.27) must hold for every choice of the velocity fields $\mathscr{U}$ and $\mathbf{t}\cdot\mathbf{u}$, it follows

$$\psi = \omega, \tag{5.28}$$

$$\sigma = \phi. \tag{5.29}$$

As a consequence, the modified expressions of configurational stresses are

$$\mathbf{P} = \psi\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}, \tag{5.30}$$

$$\boldsymbol{c} = \phi\mathbf{t} + \bar{\mu}\mathbf{n} - \boldsymbol{F}^{\mathrm{T}}T - N^{\mathrm{t}}\overset{\bullet}{*}S. \tag{5.31}$$

Expression (5.30) has been already deduced in Mariano (2000). In Eqs. (5.30) and (5.31), the terms $\nabla\varphi^{\mathrm{t}} * \mathscr{S}$ and $N^{\mathrm{t}}\overset{\bullet}{*}S$ represent the contribution of the interactions on the substructure to configurational stresses.

Note that such contributions are related *only* to the possibility of recognizing microstresses associated to the order parameters. When microstresses are absent (owing to the absence of a connection by which $\nabla\varphi$ is defined in covariant way) the above mentioned contributions disappear. This is the case of internal variable models [13] in which the thermodynamical affinities associated to the internal variables do not perform explicit working and balance equations are *not* associated to them. In this last case the Eshelby tensor is not modified ($\mathbf{P}$ reduces thus to $\psi\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T}$) and the internal variables influences only the explicit expression of the free energy.

In the general case of multifield theories, otherwise, expressions (5.29) and (5.30) holds in their complete form.

Now, some addenda of Eq. (5.29) may be grouped separately in order to simplify them. In particular, taking into account Eq. (5.30), set

$$\Lambda_1 = \sum_{i=1}^{\bar N}\int_{\Gamma_i(\ell)\cap\mathscr{B}^*(\ell)} (\mathbf{n}_i\cdot[\mathbf{P}]_{(i)}\mathbf{n}_i + \left(\phi - \boldsymbol{F}^{\mathrm{T}}T\cdot t - N^{\mathrm{t}}\overset{\bullet}{*}S\cdot t\right)_i\mathscr{k}_i + (\partial_{\mathscr{s}}\boldsymbol{c})_i\cdot\mathbf{n}_i + (\boldsymbol{c}\cdot\partial_{\mathscr{s}}\mathbf{n})_i)\mathscr{U}_i\,\mathrm{d}l. \tag{5.32}$$

[13] See the standard model of Coleman and Gurtin (1967).

Thus, considering that

$$\partial_s \mathbf{n} = -k t, \tag{5.33}$$

it follows that

$$\boldsymbol{c} \cdot \partial_s \mathbf{n} = -\left(\phi - \boldsymbol{F}^{\mathrm{T}} T \cdot t - N^{\mathrm{t}} \overset{\bullet}{*} S \cdot t\right) k. \tag{5.34}$$

Consequently, substituting Eq. (5.34) into Eq. (5.32) and taking account of the configurational balance (4.17), Λ_1 can be expressed as

$$\Lambda_1 = -\sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell) \cap \mathscr{B}^*(\ell)} (\mathsf{f} \cdot \mathbf{n})_{(i)} \mathscr{U}_i \, \mathrm{d}l. \tag{5.35}$$

To do other simplifications, consider that

$$T \cdot \mathscr{U} \langle \mathbf{F} \rangle \mathbf{n}|_{\mathscr{J}} = T \cdot \langle \mathbf{F} \rangle \mathbf{v}_{\mathscr{J}} - T \cdot \langle \mathbf{F} \rangle \mathbf{t} (\mathbf{v}_{\mathscr{J}} \cdot \mathbf{t}), \tag{5.36}$$

$$S \cdot \mathscr{U} \langle \nabla \varphi \rangle \mathbf{n}|_{\mathscr{J}} = S \cdot \langle \nabla \varphi \rangle \mathbf{v}_{\mathscr{J}} - S \cdot \langle \nabla \varphi \rangle \mathbf{t} (\mathbf{v}_{\mathscr{J}} \cdot \mathbf{t}). \tag{5.37}$$

Consequently, taking into account Eqs. (5.36), (5.37) and (5.31), the term in Eq. (5.32) associated with the junction can be written as

$$\begin{aligned} &\sum_{i=1}^{\bar{N}} (\phi_i (t_i \cdot \mathscr{U}_i) + \bar{\mu}_{(i)} \mathscr{U}_i + T_{(i)} \cdot (\langle \dot{\mathbf{x}} \rangle + \mathscr{U} \langle \mathbf{F} \rangle \mathbf{n})_i + S_{(i)} \cdot (\langle \dot{\varphi} \rangle + \mathscr{U} \langle \nabla \varphi \rangle \mathbf{n})_i) \Big|_{\mathscr{J}} \\ &\quad = \sum_{i=1}^{\bar{N}} (T_{(i)} \cdot (\langle \dot{\mathbf{x}} \rangle + \langle \mathbf{F} \rangle \mathbf{v}_{\mathscr{J}})_i + S_{(i)} \cdot (\langle \dot{\varphi} \rangle + \langle \nabla \varphi \rangle \mathbf{v}_{\mathscr{J}})_i) \big|_{\mathscr{J}} + \sum_{i=1}^{\bar{N}} (\mathbf{v}_{\mathscr{J}} \cdot \boldsymbol{c}_i) \Big|_{\mathscr{J}}. \end{aligned} \tag{5.38}$$

Consider also that, the term

$$\int_{\partial D_r} (\psi \mathscr{V} + \mathbf{Tm} \cdot \dot{\mathbf{x}} + \mathscr{S}\mathbf{m} \cdot \dot{\varphi}) \mathrm{d}l \tag{5.39}$$

may be written as

$$\int_{\partial D_r} (\mathbf{v} \cdot \mathbf{Pm} + \mathbf{Tm} \cdot (\dot{\mathbf{x}} + \mathbf{Fv}) + \mathscr{S}\mathbf{m} \cdot (\dot{\varphi} + \nabla \varphi \mathbf{v})) \mathrm{d}l. \tag{5.40}$$

Moreover, since $\mathbf{v}$ is not singular at the junction and approaches $\mathbf{v}_{\mathscr{J}}$ when $r \to 0$,

$$\lim_{r \to 0} \int_{\partial D_r} (\mathbf{v} \cdot \mathbf{Pm}) \mathrm{d}l + \sum_{i=1}^{\bar{N}} (\mathbf{v}_{\mathscr{J}} \cdot \boldsymbol{c}_i) \big|_{\mathscr{J}} = \left(\lim_{r \to 0} \int_{\partial D_r} \mathbf{Pm} \, \mathrm{d}l + \sum_{i=1}^{\bar{N}} \boldsymbol{c}_i \big|_{\mathscr{J}} \right) \cdot \mathbf{v}_{\mathscr{J}} = -\boldsymbol{d}_{\mathscr{J}} \cdot \mathbf{v}_{\mathscr{J}}. \tag{5.41}$$

As a consequence of previous developments, the mechanical dissipation inequality can be written as

$$\begin{aligned} &\int_{\mathscr{B}^*(\ell)} (\dot{\psi} - \mathbf{T} \cdot \dot{\mathbf{F}} - \mathbf{z} \cdot \dot{\varphi} - \mathscr{S} \cdot \nabla \dot{\varphi}) \mathrm{d}v + \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell) \cap \mathscr{B}^*(\ell)} (\mathsf{f} \cdot \mathbf{n})_{(i)} \mathscr{U}_i \, \mathrm{d}l - \sum_{i=1}^{\bar{N}} (T_{(i)} \cdot (\langle \dot{\mathbf{x}} \rangle + \langle \mathbf{F} \rangle \mathbf{v}_{\mathscr{J}})_i \\ &\quad + S_{(i)} \cdot (\langle \dot{\varphi} \rangle + \langle \nabla \varphi \rangle \mathbf{v}_{\mathscr{J}})_i) \Big|_{\mathscr{J}} + \boldsymbol{d}_{\mathscr{J}} \cdot \mathbf{v}_{\mathscr{J}} + \sum_{i=1}^{\bar{N}} \int_{\Gamma_i(\ell) \cap \mathscr{B}^*(\ell)} \Big(\phi^0 - T \cdot \mathbf{p}^0 - S \cdot \mathbf{q}^0 - \bar{\mu} \partial_s \mathscr{U} \\ &\quad - Z \cdot \left(\varphi^{\pm} \right)^0 \Big)_i \mathrm{d}l - \lim_{r \to 0} \int_{\partial D_r} (\mathbf{Tm} \cdot (\dot{\mathbf{x}} + \mathbf{Fv}) + \mathscr{S}\mathbf{m} \cdot (\dot{\varphi} + \nabla \varphi \mathbf{v})) \mathrm{d}l \leqslant 0. \end{aligned} \tag{5.42}$$

Inequality Eq. (5.42) can be further reduced by considering that, to within terms of the type o(r),

$$\lim_{r\to 0}\int_{\partial D_r}(\mathbf{Tm}\cdot(\dot{\mathbf{x}}+\mathbf{Fv}))\mathrm{d}l=(\langle\dot{\mathbf{x}}\rangle+\langle\mathbf{F}\rangle\mathbf{v}_{\mathscr{J}})\Big|_{\mathscr{J}}\cdot\lim_{r\to 0}\int_{\partial D_r}\mathbf{Tm}\,\mathrm{d}l, \tag{5.43}$$

$$\lim_{r\to 0}\int_{\partial D_r}(\mathscr{S}\mathbf{m}\cdot(\dot{\varphi}+\nabla\varphi\mathbf{v}))\mathrm{d}l=(\langle\dot{\varphi}\rangle+\langle\nabla\varphi\rangle\mathbf{v}_{\mathscr{J}})\Big|_{\mathscr{J}}\cdot\lim_{r\to 0}\int_{\partial D_r}\mathscr{S}\mathbf{m}\,\mathrm{d}l. \tag{5.44}$$

The proof of Eq. (5.43) can be found in Simha and Bhattacharya (1998, p. 2344); the one of Eq. (5.44) is a simple adaptation of that.

Consequently, by using the balance (3.21) and (3.22),

$$\begin{aligned}&\sum_{i=1}^{\bar N}(T_{(i)}\cdot(\langle\dot{\mathbf{x}}\rangle+\langle\mathbf{F}\rangle\mathbf{v}_{\mathscr{J}})_i+S_{(i)}\cdot(\langle\dot{\varphi}\rangle+\mathscr{U}\langle\nabla\varphi\rangle\mathbf{v}_{\mathscr{J}})_i)\Big|_{\mathscr{J}}+\lim_{r\to 0}\int_{\partial D_r}(\mathbf{Tm}\cdot(\dot{\mathbf{x}}+\mathbf{Fv})\\&\quad+\mathscr{S}\mathbf{m}\cdot(\dot{\varphi}+(\nabla\varphi)\mathbf{v}))\mathrm{d}l\\&=(\langle\dot{\mathbf{x}}\rangle+\langle\mathbf{F}\rangle\mathbf{v}_{\mathscr{J}})|_{\mathscr{J}}\cdot\left(\lim_{r\to 0}\int_{\partial D_r}\mathbf{Tm}\,\mathrm{d}l+\sum_{i=1}^{\bar N}T_{(i)}\Big|_{\mathscr{J}}\right)+(\langle\dot{\varphi}\rangle+\langle\nabla\varphi\rangle\mathbf{v}_{\mathscr{J}})|_{\mathscr{J}}\cdot\left(\lim_{r\to 0}\int_{\partial D_r}\mathscr{S}\mathbf{m}\,\mathrm{d}l\right.\\&\qquad\left.+\sum_{i=1}^{\bar N}S_{(i)}\Big|_{\mathscr{J}}\right)=0.\end{aligned} \tag{5.45}$$

Taking Eq. (5.45) into account, since Eq. (5.42) must be valid for every choice of $\mathscr{B}^*$, the following localized dissipation inequalities can be obtained, namely

$$\dot{\psi}-\mathbf{T}\cdot\dot{\mathbf{F}}-\mathbf{z}\cdot\dot{\varphi}-\mathscr{S}\cdot\nabla\dot{\varphi}\leqslant 0\qquad\text{in }\mathscr{B}, \tag{5.46}$$

$$\phi^0-T\cdot\mathbf{p}^0-S\cdot\mathbf{q}^0-Z\cdot\varphi^{\pm 0}-\bar{\mu}\partial_s\mathscr{U}+\mathsf{f}\mathscr{U}\leqslant 0\qquad\text{at }\Gamma, \tag{5.47}$$

$$\boldsymbol{d}_{\mathscr{J}}\cdot\mathbf{v}_{\mathscr{J}}\leqslant 0\qquad\text{at }\mathscr{J}. \tag{5.48}$$

Now, assume that the bulk free energy and the interfacial free energy have, respectively, the following constitutive structures:

$$\psi=\hat{\psi}(\mathbf{F},\varphi,\nabla\varphi), \tag{5.49}$$

$$\phi=\hat{\phi}(\mathbf{p},\mathbf{q},\theta,\varphi^{\pm}), \tag{5.50}$$

where θ is the limiting crystallographic orientation of the interfaces and is such that $\theta^0=\partial_s\mathscr{U}$ (Gurtin, 1993b).

By calculating the relevant time derivatives of ψ and ϕ and inserting them into Eqs. (5.46) and (5.47), the validity of Eqs. (5.46) and (5.47) for every choice of the velocity fields involved in their expressions implies

$$\mathbf{T}=\partial_{\mathbf{F}}\psi,\qquad\mathbf{z}=\partial_{\varphi}\psi,\qquad\mathscr{S}=\partial_{\nabla\varphi}\psi, \tag{5.51}$$

$$T=\partial_{\mathbf{p}}\phi,\qquad S=\partial_{\mathbf{q}}\phi,\qquad Z=\partial_{\varphi^{\pm}}\phi, \tag{5.52}$$

$$\bar{\mu}=\partial_{\theta}\phi, \tag{5.53}$$

$$-(\mathsf{f}\cdot\mathbf{n})\mathscr{U}\geqslant 0\qquad\text{at }\Gamma_i, \tag{5.54}$$

$$-\boldsymbol{d}_{\mathscr{J}}\cdot\mathbf{v}_{\mathscr{J}}\geqslant 0\qquad\text{at }\mathscr{J}. \tag{5.55}$$

Now, the driving force at the ith interface d_{Γ_i} is given by

$$d_{\Gamma_i} = -\mathsf{f}_i \cdot \mathbf{n}_i, \tag{5.56}$$

while the driving force at the junction $\boldsymbol{d}_{\mathscr{J}}$ is given by

$$\mathbf{d}_{\mathscr{J}} = -\boldsymbol{d}_{\mathscr{J}}. \tag{5.57}$$

Consequently, the use of configurational balances (4.17) and (4.18) and the results Eqs. (5.30) and (5.31) lead to

$$d_\Gamma = \mathbf{n} \cdot \left[\psi\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right]\mathbf{n} + \left(\phi - \mathbf{F}^{\mathrm{T}}T \cdot t - N^{\mathrm{t}}S \cdot t\right)\kappa + \partial_s\bar{\mu} \qquad \text{at } \Gamma_i, \tag{5.58}$$

$$\mathbf{d}_{\mathscr{J}} = \lim_{r\to 0}\int_{\partial D_r}\left(\psi\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right)\mathbf{m}\,\mathrm{d}l + \sum_{i=1}^{N}\left(\phi t + \bar{\mu}\mathbf{n} - \mathbf{F}^{\mathrm{T}}T - N^{\mathrm{t}}S\right)_i\Big|_{\mathscr{J}} \qquad \text{at } \mathscr{J}. \tag{5.59}$$

The dissipation inequalities (5.54) and (5.55) allow us to determine the evolution equations for the interfaces and the junction from Eqs. (5.58) and (5.59). Since Eqs. (5.54) and (5.55) must be valid for every choice of velocity fields $\mathscr{U}$ and $\mathbf{v}_{\mathscr{J}}$, in fact, d_Γ and $\mathbf{d}_{\mathscr{J}}$ need have structures of the form

$$d_\Gamma = d_\Gamma(\bullet)\mathscr{U}, \tag{5.60}$$

$$\mathbf{d}_{\mathscr{J}} = \mathbf{d}^*_{\mathscr{J}}\left(\bullet; \mathbf{v}_{\mathscr{J}}\right), \tag{5.61}$$

where d_Γ is a positive definite scalar function, given by a constitutive relation and $\mathbf{d}^*_{\mathscr{J}}$ is a vector function, given by a constitutive relation and such that $\mathbf{d}^*_{\mathscr{J}} \cdot \mathbf{v}_{\mathscr{J}} \geqslant 0$.

In particular, a suitable choice for $\mathbf{d}^*_{\mathscr{J}}$ is the following:

$$\mathbf{d}^*_{\mathscr{J}} = d^*_{\mathscr{J}}(\bullet)\mathbf{v}_{\mathscr{J}}, \tag{5.62}$$

where $d^*_{\mathscr{J}}$ is a positive definite scalar function given by a constitutive relation.

Possible constitutive choices for d_Γ and $d^*_{\mathscr{J}}$ are the following:

$$d_\Gamma = \hat{d}_\Gamma\left(\mathbf{p}, \mathbf{q}, \varphi^{\pm}, \theta\right), \tag{5.63}$$

$$d_{\mathscr{J}} = \hat{d}_{\mathscr{J}}\left(\mathbf{p}, \mathbf{q}, \varphi^{\pm}, \theta\right). \tag{5.64}$$

Moreover, Eqs. (5.54) and (5.55) do not exclude a priori the possibility of including in d_Γ and $d^*_{\mathscr{J}}$ rate effects or higher-order gradient effects.

Finally, by using Eqs. (5.60) and (5.62), the evolution equations for the interfaces and the junction are the following:

$$d_\Gamma\mathscr{U} = \mathbf{n} \cdot \lfloor\psi\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}\rfloor\mathbf{n} + \left(\phi - \boldsymbol{F}^{\mathrm{T}}T \cdot t - N^{\mathrm{t}}\mathbin{\overset{\bullet}{*}}S \cdot t\right)\kappa + \partial_s\bar{\mu} \quad \text{at } \Gamma_i, \tag{5.65}$$

$$d_{\mathscr{J}}\mathbf{v}_{\mathscr{J}} = \lim_{r\to 0}\int_{\partial D_r}\left(\psi\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right)\mathbf{m}\,\mathrm{d}l + \sum_{i=1}^{\bar{N}}\left(\phi t + \bar{\mu}\mathbf{n} - \boldsymbol{F}^{\mathrm{T}}T - N^{\mathrm{t}}\mathbin{\overset{\bullet}{*}}S\right)_i\Big|_{\mathscr{J}} \quad \text{at } \mathscr{J}. \tag{5.66}$$

6. Inertial effects

When inertial effects are noticeable, the relevant expression of the modified Eshelby's tensor can be obtained by considering a mechanical dissipation inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\{\text{free energy} + \text{kinetic energy}\} - W(\mathscr{B}^*) \leqslant 0. \tag{6.1}$$

Here the term "kinetic energy" is given by

$$\text{kinetic energy} = \tfrac{1}{2}\rho|\dot{\mathbf{x}}| + \chi(\varphi, \dot{\varphi}), \tag{6.2}$$

where ρ is the mass density of the bulk and χ is the contribution to the kinetic energy associated to order parameter φ. In particular $\chi(\cdot,\cdot)$ must be such that $\chi(\cdot, 0) = 0$.

By following a procedure analogous to the one used previously, it is possible to determine the dynamic counterpart of $\mathbf{P}$, namely $\mathbf{P}^{\text{dyn}}$ (see Mariano (2000), for the complete proof). $\mathbf{P}^{\text{dyn}}$ is given by

$$\mathbf{P}^{\text{dyn}} = \left(\psi + \tfrac{1}{2}\rho|\dot{\mathbf{x}}|^2 + \chi(\varphi, \dot{\varphi})\right)\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}. \tag{6.3}$$

Note that the inertial contribution to the bulk configurational stress is given only by a spherical tensor, namely [14]

$$\left(\tfrac{1}{2}\rho|\dot{\mathbf{x}}|^2 + \chi(\varphi, \dot{\varphi})\right)\mathbf{I}. \tag{6.4}$$

Moreover, if the surface configurational force f is decomposed in its inertial and non-inertial parts, f^{in} and f^{ni} respectively, f^{in} has the following expression (Mariano, 2000, Eq. (4.88)) [15]

$$\mathsf{f}^{\text{in}} = \rho(\langle\mathbf{F}\rangle[\dot{\mathbf{x}}]\mathscr{U})n + (\langle\nabla\varphi^{\mathrm{t}}\rangle \dot{*} [\partial_{\dot{\varphi}}\chi]\mathscr{U}) - [\chi(\varphi, \dot{\varphi})] \quad \text{at } \Gamma_i. \tag{6.5}$$

Now, by using Eqs. (6.3) and (6.5), Eqs. (5.57) and (5.58) can be written as follows:

$$\begin{aligned} d_\Gamma = \mathbf{n} \cdot \left[\left(\psi + \tfrac{1}{2}\rho|\dot{\mathbf{x}}|^2\right)\mathbf{I} - \boldsymbol{F}^{\mathbf{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right]\mathbf{n} + \left(\phi - \boldsymbol{F}^{\mathrm{T}}T \cdot t - N^{\mathrm{t}} \overset{\bullet}{*} S \cdot t\right)\mathscr{k} + \partial_s\bar{\mu} + \rho(\langle\mathbf{F}\rangle[\dot{\mathbf{x}}]\mathscr{U}) \cdot \mathbf{n} \\ + (\langle\nabla\varphi^{\mathrm{t}}\rangle \dot{*} [\partial_{\dot{\varphi}}\chi]\mathscr{U}) \cdot \mathbf{n}, \end{aligned} \tag{6.6}$$

$$\mathbf{d}_{\mathscr{J}} = \lim_{r\to 0} \int_{\partial D_r} \left(\left(\psi + \tfrac{1}{2}\rho|\dot{\mathbf{x}}|^2 + \chi(\varphi, \dot{\varphi})\right)\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{T} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}\right)\mathbf{m}\,\mathrm{d}l + \sum_{i=1}^{\bar{N}} (\phi t + \bar{\mu}\mathbf{n} - \boldsymbol{F}^{\mathrm{T}}T - N^{\mathrm{t}} \overset{\bullet}{*} S)_i\Big|_{\mathscr{J}}. \tag{6.7}$$

The use of Eqs. (5.60) and (5.62) into Eqs. (6.6) and (6.7) leads to the relevant kinetic equations when inertial effects are noticeable.

7. Neglecting deformations

When the macroscopic deformations are negligible, Eqs. (5.58) and (5.59) reduce to

$$d_\Gamma = \mathbf{n} \cdot [\psi\mathbf{I} - \nabla\varphi^{\mathrm{t}} * \mathscr{S}]\mathbf{n} + \left(\phi - N^{\mathrm{t}} \overset{\bullet}{*} S \cdot t\right)\mathscr{k} + \partial_s(\mu - \langle\nabla\varphi\rangle\mathbf{n} \cdot S) \quad \text{at } \Gamma_i, \tag{7.1}$$

$$\mathbf{d}_{\mathscr{J}} = \lim_{r\to 0} \int_{\partial D_r} (\psi\mathbf{I} - \nabla\varphi^{\mathrm{t}} * \mathscr{S})\mathbf{m}\,\mathrm{d}l + \sum_{i=1}^{\bar{N}} \left(\phi t + (\mu - \langle\nabla\varphi\rangle\mathbf{n} \cdot S) - N^{\mathrm{t}} \overset{\bullet}{*} S\right)_i\Big|_{\mathscr{J}} \quad \text{at } \mathscr{J}. \tag{7.2}$$

[14] Other inertial effects can be recognized in Eq. (4.16) where the inertial part of the external configurational force e is given by $e^{\text{in}} = \rho\mathbf{F}^{\mathrm{T}}\ddot{\mathbf{x}} + \nabla\varphi^{\mathrm{t}} \overset{\bullet}{*} \{\overset{\bullet}{\overline{(\partial_{\dot{\varphi}}\chi)}} - \partial_\varphi\chi\}$ (Mariano, 2000, Eq. (3.38)).

[15] Here the Eq. (4.88) in Mariano (2000) is emended of a trivial print error in its last two lines.

Note that the presence of microstructure within the body implies an articulate behavior of the interfaces and the junction, even in absence of macroscopic deformations.

As a special case, consider a situation in which not only macroscopic deformations are neglected but also

- the order parameter is a scalar quantity (say it represents the volume fraction of a phase in some two phase material),
- interfacial structure is neglected,
- the bulk Helmoltz free energy ψ is Ginzburg–Landau's-like, more precisely

$$\psi(\varphi, \nabla\varphi) = \tfrac{1}{2}b|\nabla\varphi|^2 + \sigma(\varphi). \tag{7.3}$$

In Eq. (7.3), $\sigma(\bullet)$ is a double well coarse grained potential and b a material constant. The simplest expression of $\sigma(\bullet)$ is given by

$$\sigma(\varphi) = a\left(1 - \varphi^2\right)^2 \tag{7.4}$$

with a some constant.

In this case, T, S, Z, $\bar{\mu}$, ϕ, $\mathbf{T}$ vanish identically, the bulk microstress is a vector and is given by

$$\mathscr{S} = b\nabla\varphi, \tag{7.5}$$

while the internal force $\mathbf{z}$ is scalar, namely

$$\mathbf{z} = 4a\varphi\left(\varphi^2 - 1\right). \tag{7.6}$$

Finally, the driving forces are given by

$$d_\Gamma = \mathbf{n} \cdot \left[\left(\tfrac{1}{2}b|\nabla\varphi|^2 + \sigma(\varphi)\right)\mathbf{I} - b\nabla\varphi \otimes \nabla\varphi\right]\mathbf{n}, \tag{7.7}$$

$$\mathbf{d}_\mathscr{J} = \lim_{r\to 0}\int_{\partial D_r}\left(\left(\tfrac{1}{2}b|\nabla\varphi|^2 + \sigma(\varphi)\right)\mathbf{I} - b\nabla\varphi \otimes \nabla\varphi\right)\mathbf{m}\,\mathrm{d}l. \tag{7.8}$$

Note that the driving force at the junction is given by quantity having an expression analogous to the one of Rice's integral.

In this case,

$$d_\Gamma = \hat{d}_\Gamma\left(\mathbf{q}, \varphi^\pm, \theta\right), \tag{7.9}$$

$$d_\mathscr{J} = \hat{d}_\mathscr{J}\left(\mathbf{q}, \varphi^\pm, \theta\right). \tag{7.10}$$

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